MICRO-LOCAL ANALYSIS IN SOME SPACES OF ULTRADISTRIBUTIONS

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Abstract. In this paper we extend some results from [16] and [21], concerning wave-front sets of Fourier-Lebesgue and modulation space types, to a broader class of spaces of ultradistributions, and relate these wave-front sets with the usual wave-front sets of ultradistributions.

Furthermore, we use Gabor frames for the description of discrete wave-front sets, and prove that these wave-front sets coincide with corresponding continuous ones.

0. Introduction

Wave-front sets with respect to Fourier Lebesgue and modulation spaces were introduced in [21] and studied further in [20,22,23]. Among other properties, it was proved that wave-front sets of Fourier Lebesgue and modulation spaces agree with each others, and that the usual wave-front sets with respect to smoothness (cf. [15, Sections 8.1–8.3]) can be obtained as wave-front sets of sequences of Fourier Lebesgue or modulation spaces. Discrete versions of wave-front sets of Fourier Lebesgue and modulation spaces, related to those wave-front sets in [26], were introduced and studied in [16]. In particular, it was proved that these agree with corresponding continuous ones.

In this paper we extend some results from [16] and [21] to a broader class of spaces of ultradistributions. Instead of polynomial growth at infinity, here we study objects which may have almost exponential growth at infinity. We may thereby recover the usual wave-front sets of ultradistributions, see Section 2.

Furthermore, we introduce discrete versions of wave-front sets of ultradistributions and prove that these wave-front sets coincide with corresponding continuous ones (cf. [16]). The results in form of series, established when introducing these discrete versions, might be useful for numerical analysis of micro-local properties of functions and ultradistributions. For example, we use Gabor frames for the description of discrete wave-front sets. (See [8, 9] for numerical treatment of Gabor frame theory.) With that respect we emphasize that in the process of

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analysis and synthesis of a signal, Gabor frame coefficients also give information on micro-local properties of the signal.

Since compactly supported smooth functions are used in the process of microlocalization we are limited to the use of weights with almost exponential growth at infinity described by the Beurling-Domar condition. We refer to subsection 0.1 for the notions and to [12] for a discussion on the role of weights in time-frequency analysis.

Our investigation can therefore be considered as the starting point in the study of analytic wave-front sets and pseudodifferential operators with ultrapolynomial symbols, known also as symbol-global type operators. This will be done in a separate paper.

The paper is organized as follows. In Section 1 we introduce wave-front sets of Fourier Lebesgue types with respect to ultradistributions. These are compared to other types of wave-front sets in Section 2. In Section 3 we discuss wave-front sets of modulation space types and show equivalence with those of Fourier Lebesgue types. Sections 4–6 are devoted to discrete versions of wave-front sets. For the reader's convenience the results in Sections 1–6 are formulated in terms of ultradistributions of Roumieu type and necessary explanations concerning the Beurling type ultradistributions and differences between the Roumieu and Beurling cases are given in the last section.

0.1. **Basic notions and notation.** In this subsection we collect some notation and notions which will be used in the sequel.

We put $\mathbf{N} = \{0, 1, 2, \dots\}$, $\mathbf{Z}_+ = \{1, 2, 3, \dots\}$, $\langle x \rangle = (1 + |x|^2)^{1/2}$, for $x \in \mathbf{R}^d$, and $A \lesssim B$ to indicate $A \le cB$ for a suitable constant $c > 0$. The symbol $B_1 \hookrightarrow B_2$ denotes the continuous and dense embedding of the topological vector space B_1 into B_2. The scalar product in L^2 is denoted by $(\cdot, \cdot)_{L^2}$. Translation and modulation operators are given by

$$T_x f(t) = f(t - x) \quad \text{and} \quad M_\xi f(t) = e^{i\langle \xi, t\rangle} f(t)\,. \tag{0.1}$$

0.1.1. *Weights.* In general, a weight function is a non-negative function.

Definition 0.1. Let ω, v be non-negative functions. Then

(1) v is called *submultiplicative* if

$$v(x + y) \le v(x)v(y), \qquad \forall\, x, y \in \mathbf{R}^d;$$

(2) ω is called *v-moderate* if

$$\omega(x + y) \lesssim v(x)\omega(y), \qquad \forall\, x, y \in \mathbf{R}^d.$$

For a given submultiplicative weight v the set of all v-moderate weights will be denoted by $\mathscr{M}_v$.

If $\omega \in \mathscr{M}_v$, then $1/v \lesssim \omega \lesssim v$, $\omega \ne 0$ everywhere and $1/\omega \in \mathscr{M}_v$.

In the sequel v will always stand for a submultiplicative function. Submultiplicativity implies that v is dominated by an exponential function, i.e.

$$\exists\, C, k > 0 \quad \text{such that} \quad v \le Ce^{k|\,\cdot\,|}. \tag{0.2}$$

A submultiplicative weight v satisfies the GRS condition (the Gelfand-Raikov-Shilov condition) if $\lim_{n\to\infty} v(nx)^{1/n} = 1$, for every $x \in \mathbf{R}^d$.

Let $s > 1$. By $\mathscr{M}_{\{s\}}(\mathbf{R}^d)$ we denote the set of all weights which are moderate with respect to a weight v which satisfies $v \le Ce^{k|\,\cdot\,|^{1/s}}$ for some positive constants C and k. The weight v satisfy the Beurling-Domar non-quasi-analyticity condition which takes the form

$$\sum_{n=0}^{\infty} \frac{\log v(nx)}{n^2} < \infty, \quad x \in \mathbf{R}^d,$$

and which is stronger than the Gelfand-Raikov-Shilov condition, cf. [12].

0.1.2. *Test function spaces and their duals.* Next we introduce spaces of test functions and their duals in the context of spaces of ultradistributions. We start with Gelfand-Shilov type spaces.

Definition 0.2. Let $s > 1$ and $A > 0$. We denote by $\mathcal{S}^s_A(\mathbf{R}^d)$ the space of all functions $\varphi \in C^\infty(\mathbf{R}^d)$ such that the norm

$$\|\varphi\|_{s,A} = \sup_{\alpha,\beta\in\mathbf{N}_0^d} \sup_{x\in\mathbf{R}^d} \frac{A^{|\alpha+\beta|}}{\alpha!^s\beta!^s} \langle x\rangle^{|\alpha|} |\varphi^{(\beta)}(x)|$$

is finite. Then the projective limit is denoted by $\mathcal{S}^{(s)}(\mathbf{R}^d)$, i. e.,

$$\mathcal{S}^{(s)}(\mathbf{R}^d) = \text{proj} \lim_{A\to\infty} \mathcal{S}^s_A(\mathbf{R}^d),$$

and the inductive limit is denoted by $\mathcal{S}^{\{s\}}(\mathbf{R}^d)$, i. e.,

$$\mathcal{S}^{\{s\}}(\mathbf{R}^d) = \text{ind} \lim_{A\to 0} \mathcal{S}^s_A(\mathbf{R}^d).$$

The space $\mathcal{S}^{\{s\}}$ is called the Gelfand-Shilov space of order s.

The strong dual spaces of $\mathcal{S}^{(s)}(\mathbf{R}^d)$ and $\mathcal{S}^{\{s\}}(\mathbf{R}^d)$ are spaces of tempered ultradistributions of Beurling and Roumieu type denoted by $(\mathcal{S}^{(s)})'(\mathbf{R}^d)$ and $(\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$, respectively. If $s > t$, then

$$\begin{aligned} \mathcal{S}^{(t)}(\mathbf{R}^d) &\hookrightarrow \mathcal{S}^{\{t\}}(\mathbf{R}^d) \hookrightarrow \mathcal{S}^{(s)}(\mathbf{R}^d) \hookrightarrow \mathcal{S}^{\{s\}}(\mathbf{R}^d) \\ &\hookrightarrow (\mathcal{S}^{\{s\}})'(\mathbf{R}^d) \hookrightarrow (\mathcal{S}^{(s)})'(\mathbf{R}^d) \hookrightarrow (\mathcal{S}^{\{t\}})'(\mathbf{R}^d) \hookrightarrow (\mathcal{S}^{(t)})'(\mathbf{R}^d). \end{aligned}$$

In order to perform (micro)local analysis we use the following test function spaces on open sets, cf. [18].

Definition 0.3. Let X be an open set in $\mathbf{R}^d$. For a given compact set $K \subset X$, $s > 1$ and $A > 0$ we denote by $\mathcal{E}^s_{A,K}$ the space of all $\varphi \in C^\infty(X)$ such that the norm

$$\|\varphi\|_{s,A,K} = \sup_{\beta\in\mathbf{N}_0^n} \sup_{x\in K} \frac{A^{|\beta|}}{\beta!^s} |\varphi^{(\beta)}(x)| \tag{0.3}$$

is finite.

We denote by $\mathcal{E}^s_A(K)$ the space of functions $\varphi \in C^\infty(X)$ such that (0.3) holds and $\operatorname{supp}\varphi \subseteq K$.

Let $(K_n)_n$ be a sequence of compact sets such that $K_n \subset\subset K_{n+1}$ and $\bigcup K_n = X$. Then

$$\begin{aligned}
\mathcal{E}^{(s)}(X) &= \text{proj} \lim_{n\to\infty} (\text{proj} \lim_{A\to\infty} \mathcal{E}^s_{A,K_n}),\\
\mathcal{E}^{\{s\}}(X) &= \text{proj} \lim_{n\to\infty} (\text{ind} \lim_{A\to 0} \mathcal{E}^s_{A,K_n}),\\
\mathcal{D}^{(s)}(X) &= \text{ind} \lim_{n\to\infty} (\text{proj} \lim_{A\to\infty} \mathcal{E}^s_{A}(K_n)),
\end{aligned}$$

and

$$\mathcal{D}^{\{s\}}(X) = \text{ind} \lim_{n\to\infty} (\text{ind} \lim_{A\to 0} \mathcal{E}^s_{A}(K_n)).$$

The spaces of linear functionals over $\mathcal{D}^{(s)}(X)$ and $\mathcal{D}^{\{s\}}(X)$, denoted by $(\mathcal{D}^{(s)})'(X)$ and $(\mathcal{D}^{\{s\}})'(X)$ respectively, are called the spaces of *ultradistributions* of Beurling and Roumieu type respectively, while the spaces of linear functionals over $\mathcal{E}^{(s)}(X)$ and $\mathcal{E}^{\{s\}}(X)$, denoted by $(\mathcal{E}^{(s)})'(X)$ and $(\mathcal{E}^{\{s\}})'(X)$, respectively are called the spaces of *ultra-distributions of compact support* of Beurling and Roumieu type respectively, [18]. We have

$$\begin{aligned}
&(\mathcal{E}^{\{s\}})'(X) \subset (\mathcal{E}^{(s)})'(X), \quad (\mathcal{E}^{(s)})'(X) \subset (\mathcal{E}^{(s)})'(\mathbf{R}^d) \quad \text{and}\\
&(\mathcal{E}^{\{s\}})'(X) \subset (\mathcal{E}^{\{s\}})'(\mathbf{R}^d).
\end{aligned}$$

Clearly,

$$(\mathcal{E}^{\{s\}})'(\mathbf{R}^d) \subset (\mathcal{S}^{\{s\}})'(\mathbf{R}^d) \subset (\mathcal{D}^{\{s\}})'(\mathbf{R}^d)$$

and

$$(\mathcal{E}^{(s)})'(\mathbf{R}^d) \subset (\mathcal{S}^{(s)})'(\mathbf{R}^d) \subset (\mathcal{D}^{(s)})'(\mathbf{R}^d).$$

Any ultra-distribution with compact support can be viewed as an element of $(\mathcal{S}^{(1)})'(\mathbf{R}^d)$.

Obviously, $\mathcal{D}^{(s)}(X)$ ($\mathcal{D}^{\{s\}}(X)$ resp.) are subspaces of $\mathcal{E}^{(s)}(X)$ (of $\mathcal{E}^{\{s\}}(X)$ resp.) whose elements are compactly supported. We also remark that a usual notation for the space $\mathcal{D}^{\{s\}}(X)$ is $G^s(X)$ (cf. [24]).

0.1.3. *Fourier-Lebesgue spaces.* The Fourier transform $\mathscr{F}$ is the linear and continuous mapping on $\mathscr{S}'(\mathbf{R}^d)$ which takes the form

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(x) e^{-i\langle x,\xi\rangle}\, dx$$

when $f \in L^1(\mathbf{R}^d)$. It is a homeomorphism on $(\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$ (on $(\mathcal{S}^{(s)})'(\mathbf{R}^d)$ resp.) which restricts to a homeomorphism on $\mathcal{S}^{\{s\}}(\mathbf{R}^d)$ (on $\mathcal{S}^{(s)}(\mathbf{R}^d)$ resp.) and to a unitary operator on $L^2(\mathbf{R}^d)$.

Let $q \in [1,\infty]$, $s > 1$ and $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^d)$. The (weighted) Fourier Lebesgue space $\mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$ is the inverse Fourier image of $L^q_{(\omega)}(\mathbf{R}^d)$, i. e. $\mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$ consists of all $f \in (\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$ such that

$$\|f\|_{\mathscr{F}L^q_{(\omega)}} \equiv \|\widehat{f} \cdot \omega\|_{L^q}. \tag{0.4}$$

is finite. If $\omega = 1$, then the notation $\mathscr{F}L^q$ is used instead of $\mathscr{F}L^q_{(\omega)}$. We note that if $\omega(\xi) = \langle\xi\rangle^s$, then $\mathscr{F}L^q_{(\omega)}$ is the Fourier image of the Bessel potential space H^p_s.

Remark 0.4. In many situations it is convenient to permit an x dependency for the weight ω in the definition of Fourier Lebesgue spaces. More precisely, for each $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$ we let $\mathscr{F}L^q_{(\omega)}$ be the set of all ultradistributions f such that

$$\|f\|_{\mathscr{F}L^q_{(\omega)}} \equiv \|\widehat{f}\,\omega(x,\,\cdot\,)\|_{L^q}$$

is finite. Since ω is v-moderate it follows that different choices of x give rise to equivalent norms. Therefore the condition $\|f\|_{\mathscr{F}L^q_{(\omega)}} < \infty$ is independent of x, and it follows that $\mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$ is independent of x although $\|\cdot\|_{\mathscr{F}L^q_{(\omega)}}$ might depend on x.

1. Wave-front sets of Fourier-Lebesgue type in spaces of Roumieu type ultradistributions

In this section we introduce wave-front sets of Fourier-Lebesgue type in spaces of ultradistributions of Roumieu type.

Let $s > 1$, $q \in [1,\infty]$, and $\Gamma \subseteq \mathbf{R}^d \setminus 0$ be an open cone. If $f \in (\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$ and $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$ we define

$$|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega)}} = |f|_{\mathscr{F}L^{q,\Gamma}_{(\omega),x}} \equiv \Big(\int_\Gamma |\widehat{f}(\xi)\omega(x,\xi)|^q\, d\xi \Big)^{1/q} \tag{1.1}$$

(with obvious interpretation when $q = \infty$). We note that $|\cdot|_{\mathscr{F}L^{q,\Gamma}_{(\omega),x}}$ defines a semi-norm on $(\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$ which might attain the value $+\infty$. Since ω is v-moderate it follows that different $x \in \mathbf{R}^d$ gives rise to equivalent semi-norms $|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega),x}}$. Furthermore, if $\Gamma = \mathbf{R}^d \setminus 0$, $f \in \mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$ and $q < \infty$, then $|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega),x}}$ agrees with the Fourier Lebesgue norm $\|f\|_{\mathscr{F}L^q_{(\omega),x}}$ of f.

For the sake of notational convenience we set

$$\mathcal{B} = \mathscr{F}L^q_{(\omega)} = \mathscr{F}L^q_{(\omega)}(\mathbf{R}^d), \quad \text{and} \quad |\,\cdot\,|_{\mathcal{B}(\Gamma)} = |\,\cdot\,|_{\mathscr{F}L^{q,\Gamma}_{(\omega),x}}. \tag{1.2}$$

We let $\Theta_{\mathcal{B}}(f) = \Theta_{\mathscr{F}L^q_{(\omega)}}(f)$ be the set of all $\xi \in \mathbf{R}^d \setminus 0$ such that $|f|_{\mathcal{B}(\Gamma)} < \infty$, for some open conical neighbourhood $\Gamma = \Gamma_\xi$ of ξ. We also let $\Sigma_{\mathcal{B}}(f)$ be the complement of $\Theta_{\mathcal{B}}(f)$ in $\mathbf{R}^d \setminus 0$. Then $\Theta_{\mathcal{B}}(f)$ and $\Sigma_{\mathcal{B}}(f)$ are open respectively closed subsets in $\mathbf{R}^d \setminus 0$, which are independent of the choice of $x \in \mathbf{R}^d$ in (1.1).

Definition 1.1. Let $s > 1$, $q \in [1, \infty]$, $\mathcal{B}$ be as in (1.2), and let X be an open subset of $\mathbf{R}^d$. If $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, then the wave-front set of ultradistribution $f \in (\mathcal{D}^{(s)})'(\mathbf{R}^d)$, $\mathrm{WF}_{\mathcal{B}}(f) \equiv \mathrm{WF}_{\mathscr{F}L^q_{(\omega)}}(f)$ with respect to $\mathcal{B}$ consists of all pairs (x_0, ξ_0) in $X \times (\mathbf{R}^d \setminus 0)$ such that $\xi_0 \in \Sigma_{\mathcal{B}}(\varphi f)$ holds for each $\varphi \in \mathcal{D}^{(s)}(X)$ such that $\varphi(x_0) \neq 0$.

We note that $\mathrm{WF}_{\mathcal{B}}(f)$ is a closed set in $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, since it is obvious that its complement is open. We also note that if $x \in \mathbf{R}^d$ is fixed and $\omega_0(\xi) = \omega(x, \xi)$, then $\mathrm{WF}_{\mathcal{B}}(f) = \mathrm{WF}_{\mathscr{F}L^q_{(\omega_0)}}(f)$, since $\Sigma_{\mathcal{B}}$ is independent of x.

The following theorem shows that wave-front sets with respect to $\mathscr{F}L^q_{(\omega)}$ satisfy appropriate micro-local properties. It also shows that such wave-front sets are decreasing with respect to the parameter q, and increasing with respect to the weight ω.

Theorem 1.2. *Let $s > 1$, $q, r \in [1, \infty]$, X be an open set in $\mathbf{R}^d$ and $\omega, \vartheta \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$ be such that*

$$r \le q, \quad \textit{and} \quad \omega(x, \xi) \lesssim \vartheta(x, \xi). \tag{1.3}$$

Also let $\mathcal{B}$ be as in (1.2) *and put $\mathcal{B}_0 = \mathscr{F}L^r_{(\vartheta)}(\mathbf{R}^d)$. If $f \in (\mathcal{D}^{(s)})'(\mathbf{R}^d)$ and $\varphi \in \mathcal{D}^{(s)}(X)$ then*

$$\mathrm{WF}_{\mathcal{B}}(\varphi\, f) \subseteq \mathrm{WF}_{\mathcal{B}_0}(f).$$

Proof. By the definitions it is sufficient to prove

$$\Sigma_{\mathcal{B}}(\varphi f) \subseteq \Sigma_{\mathcal{B}_0}(f) \tag{1.4}$$

when $\varphi \in \mathcal{D}^{(s)}(X)$, $\vartheta = \omega$, and $f \in (\mathcal{E}^{(s)})'(\mathbf{R}^d)$, since the statement only involves local assertions. For the same reasons we may assume that $\omega(x, \xi) = \omega(\xi)$ is independent of x. Finally, we prove the assertion for $r \in [1, \infty)$. The case $r = \infty$ follows by similar arguments and is left to the reader.

Choose open cones Γ_1 and Γ_2 in $\mathbf{R}^d$ such that $\overline{\Gamma_2} \subseteq \Gamma_1$. We will use the fact that if $f \in (\mathcal{E}^{(s)})'(\mathbf{R}^d)$ then $|\widehat{f}(\xi)\omega(\xi)| \lesssim e^{k|\xi|^{1/s}}$ for some $k > 0$

and prove that for every $N>0$, there exist $C_N>0$ such that

$$|\varphi f|_{\mathcal B(\Gamma_2)} \le C_N\Big(|f|_{\mathcal B_0(\Gamma_1)} + \sup_{\xi\in\mathbf R^d} \big(|\widehat f(\xi)\omega(\xi)|e^{-N|\xi|^{1/s}}\big)\Big) \qquad \text{when}\quad \overline{\Gamma}_2 \subseteq \Gamma_1. \tag{1.5}$$

Since $\omega \in \mathscr M_{\{s\}}(\mathbf R^{2d})$ by letting $F(\xi) = |\widehat f(\xi)\omega(\xi)|$ and $\psi(\xi) = |\widehat\varphi(\xi)v(\xi)|$ we have

$$|\varphi f|_{\mathcal B(\Gamma_2)} = \Big(\int_{\Gamma_2} |\mathscr F(\varphi f)(\xi)\omega(\xi)|^r\, d\xi\Big)^{1/r}$$
$$\lesssim \Big(\int_{\Gamma_2} \Big(\int_{\mathbf R^d} \psi(\xi-\eta)F(\eta)\, d\eta\Big)^r d\xi\Big)^{1/r} \lesssim J_1+J_2,$$

where

$$J_1 = \Big(\int_{\Gamma_2} \Big(\int_{\Gamma_1} \psi(\xi-\eta)F(\eta)\, d\eta\Big)^r d\xi\Big)^{1/r},$$
$$J_2 = \Big(\int_{\Gamma_2} \Big(\int_{\complement\Gamma_1} \psi(\xi-\eta)F(\eta)\, d\eta\Big)^r d\xi\Big)^{1/r}.$$

Let q_0 be chosen such that $1/q_0+1/q = 1+1/r$, and let χ_{Γ_1} be the characteristic function of Γ_1. Then Young's inequality gives

$$J_1 \le \Big(\int_{\mathbf R^d} \Big(\int_{\Gamma_1} \psi(\xi-\eta)F(\eta)\, d\eta\Big)^r d\xi\Big)^{1/r}$$
$$= \|\psi*(\chi_{\Gamma_1}F)\|_{L^r} \le \|\psi\|_{L^{q_0}}\|\chi_{\Gamma_1}F\|_{L^q} = C_\psi |f|_{\mathcal B_0(\Gamma_1)},$$

where $C_\psi = \|\psi\|_{L^{q_0}}<\infty$. If $\varphi\in\mathcal D^{(s)}(X)$, then for every $N>0$ there exist $C_N>0$ such that

$$\psi(\xi) = |\widehat\varphi(\xi)v(\xi)| \le C_N e^{-(N+k)|\xi|^{1/s}}e^{k|\xi|^{1/s}} \le C_N e^{-N|\xi|^{1/s}}. \tag{1.6}$$

In order to estimate J_2, we note that $\overline{\Gamma_2}\subseteq\Gamma_1$ implies that

$$|\xi-\eta|^{1/s} > c\max(|\xi|^{1/s},|\eta|^{1/s})$$
$$\ge \frac c2 (|\xi|^{1/s}+|\eta|^{1/s}),\qquad \xi\in\Gamma_2,\ \eta\notin\Gamma_1 \tag{1.7}$$

holds for some constant $c>0$, since this is true when $1=|\xi|\ge|\eta|$. A combination of (1.6) and (1.7) implies that for every $N_1>0$ we have

$$\psi(\xi-\eta)\lesssim Ce^{-2N_1(|\xi|^{1/s}+|\eta|^{1/s})}.$$

This gives

$$J_2 \lesssim \Big(\int_{\Gamma_2} \Big(\int_{\complement\Gamma_1} e^{-2N_1(|\xi|^{1/s}+|\eta|^{1/s})} F(\eta)\, d\eta \Big)^r \, d\xi \Big)^{1/r}$$

$$\lesssim \Big(\int_{\Gamma_2} \Big(\int_{\complement\Gamma_1} e^{-2N_1(|\xi|^{1/s}+|\eta|^{1/s})} e^{N_1|\eta|^{1/s}} (e^{-N_1|\eta|^{1/s}} F(\eta))\, d\eta \Big)^r \, d\xi \Big)^{1/r}$$

$$\lesssim \sup_{\eta\in \mathbf R^d} |e^{-N_1|\eta|^{1/s}} F(\eta))|,$$

which proves (1.5) and the result follows. □

2. Comparisons to other types of wave-front sets

Let $\omega \in \mathscr M_{\{s\}}(\mathbf R^{2d})$ be moderated with respect to a weight of polynomial growth at infinity and let $f \in \mathcal D'(X)$. Then $\mathrm{WF}_{\mathscr FL^q_{(\omega)}}(f)$ in Definition 1.1 is the same as the wave-front set introduced in [21, Definition 3.1]. Therefore, the information on regularity in the background of wave-front sets of Fourier-Lebesgue type in Definition 1.1 might be compared to the information obtained from the classical wave-front sets, cf. Example 4.9 in [21].

Next we compare the wave-front sets introduced in Definition 1.1 to the wave-front sets in spaces of ultradistributions given in [14, 19, 24].

Let $s > 1$ and let X be an open subset of $\mathbf R^d$. The ultradistribution $f \in (\mathcal D^{(s)})'(\mathbf R^d)$ is (s)-micro-regular ($\{s\}$-micro-regular) at (x_0, ξ_0) if there exists $\varphi \in \mathcal D^{(s)}(X)$ ($\varphi \in \mathcal D^{\{s\}}(X)$) such that $\varphi(x) = 1$ in a neighborhood of x_0 and an open cone Γ which contains ξ_0 such that for each $k > 0$ (for some $k > 0$) there is a $C > 0$ such that

$$|\widehat{\varphi f}(\xi)| \le C e^{-k|\xi|^{1/s}}, \quad \xi \in \Gamma. \tag{2.1}$$

The (s)-wave-front set ($\{s\}$-wave-front set) of f, $\mathrm{WF}_{(s)}(f)$ ($\mathrm{WF}_{\{s\}}(f)$) is defined as the complement in $X \times \mathbf R^d \setminus 0$ of the set of all (x_0, ξ_0) where f is (s)-micro-regular ($\{s\}$-micro-regular), cf. [24, Definition 1.7.1].

The $\{s\}$-wave-front set $\mathrm{WF}_{\{s\}}(f)$ can be found in [19] and agrees with certain wave-front set $\mathrm{WF}_L(f)$ introduced in [14, Chapter 8.4].

Remark 2.1. Let $s > 1$, $f \in (\mathcal D^{\{s\}})'(\mathbf R^d)$, $\varphi \in \mathcal E^{\{s\}}(\mathbf R^d)$ and $\varphi_0 \in \mathcal D^{(s)}(X)$ be such that $\varphi(x) = 1$ in a neighborhood $\operatorname{supp}\varphi_0$. Also let Γ_0, Γ be open cones such that $\overline{\Gamma_0} \subseteq \Gamma$. If (2.1) holds for some $k, C > 0$, then it follows by straight-forward computations, using similar arguments as in the proof of Theorem 1.2 that (2.1) is still true for some $k, C > 0$ after φ has been replaced by φ_0. Hence it follows that the following conditions are equivalent:

1. $(x_0, \xi_0) \notin \mathrm{WF}_{\{s\}}(f)$;
2. for some $\varphi \in \mathcal D^{\{s\}}(X)$, some conical neighborhood Γ of ξ such that $\varphi(x_0) = 1$ in a neighborhood of x_0 and some $C, k > 0$, it follows that (2.1) holds;

(3) for some $\varphi \in \mathcal{D}^{(s)}(X)$, some conical neighborhood Γ of ξ such that $\varphi(x_0) = 1$ in a neighborhood of x_0 and some $C, k > 0$, it follows that (2.1) holds.

Consequently we may always choose φ in $\mathcal{D}^{(s)}(X)$ in the definition of $\mathrm{WF}_{\{s\}}(f)$.

In most of our considerations we are concerned with $\{s\}$-micro-regularity. For this reason we set $\mathrm{WF}_s(f) = \mathrm{WF}_{\{s\}}(f)$ when $f \in (\mathcal{D}^{(s)})'$.

Proposition 2.2. *Let $q \in [1, \infty]$, $s > 1$, and let $\omega_\varepsilon(\xi) \equiv e^{k|\xi|^{1/s}}$ for $\xi \in \mathbf{R}^d$ and $k > 0$. If $f \in (\mathcal{D}^{(s)})'(\mathbf{R}^d)$ then*

$$\bigcap_{k>0} \mathrm{WF}_{\mathscr{F}L^q_{(\omega_k)}}(f) = \mathrm{WF}_s(f). \tag{2.2}$$

Proof. Recall that when k is fixed, the set $\mathrm{WF}_{\mathscr{F}L^q_{(\omega_k)}}(f)$ is defined via $\varphi \in \mathcal{D}^{(s)}(X)$, cf. Definition 1.1.

Therefore the set $\bigcap_{k>0} \mathrm{WF}_{\mathscr{F}L^q_{(\omega_k)}}(f)$ is the complement of the set of points $(x_0, \xi_0) \in \mathbf{R}^{2d}$ for which there exists $k > 0$, $\varphi \in \mathcal{D}^{(s)}(X)$ such that $\varphi(x) = 1$ in a neighborhood of x_0 and an open cone Γ which contains ξ_0 such that

$$\Big(\int_\Gamma |\widehat{\varphi f}(\xi) e^{k|\xi|^{1/s}}|^q \, d\xi \Big)^{1/q} < \infty. \tag{2.3}$$

The assertion is obviously true when $q = \infty$.

Therefore, let $q \in [1, \infty)$ and assume that $(x_0, \xi_0) \notin \mathrm{WF}_{\mathscr{F}L^\infty_{(\omega_k)}}(f)$ for some $k > 0$. Then for any $\varepsilon > 0$ such that $k - \varepsilon > 0$ we have

$$\int_\Gamma |\widehat{\varphi f}(\xi) \omega_{k-\varepsilon}(\xi)|^q \, d\xi \le \sup_{\xi \in \Gamma} \big(|\widehat{\varphi f}(\xi)|^q e^{kq|\xi|^{1/s}} \big) \int_\Gamma e^{-\varepsilon|\xi|^{1/s}} \, d\xi < \infty,$$

which means that

$$(x_0, \xi_0) \notin \bigcap_{k>0} \mathrm{WF}_{\mathscr{F}L^q_{(\omega_k)}}(f)$$

when

$$(x_0, \xi_0) \notin \bigcap_{k>0} \mathrm{WF}_{\mathscr{F}L^\infty_{(\omega_k)}}(f).$$

On the other hand, since the wave-front $\mathrm{WF}_{\mathscr{F}L^q_{(\omega)}}(f)$ is decreasing with respect to the parameter q, see Theorem 1.2, we have

$$\bigcap_{k>0} \mathrm{WF}_{\mathscr{F}L^\infty_{(\omega_k)}}(f) \subseteq \bigcap_{k>0} \mathrm{WF}_{\mathscr{F}L^q_{(\omega_k)}}(f), \quad q \in [1, \infty].$$

This completes the proof. $\square$

3. Invariance properties of wave-front sets with respect to modulation spaces

In this section we define wave-front sets with respect to modulation spaces, and show that they coincide with wave-front sets of Fourier Lebesgue types.

3.1. **Modulation spaces.** In this subsection we consider properties of modulation spaces which will be used in microlocal analysis of ultra-distributions.

Let $s > 1$. For a fixed non-zero window $\phi \in \mathcal{S}^{\{s\}}(\mathbf{R}^d)$ ($\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d)$ respectively) the short-time Fourier transform (STFT) of $f \in \mathcal{S}^{\{s\}}(\mathbf{R}^d)$ ($f \in \mathcal{S}^{(s)}(\mathbf{R}^d)$ respectively) with respect to the window ϕ is given by

$$V_\phi f(x,\xi) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(y)\, \overline{\phi(y-x)}\, e^{-i\langle \xi, y\rangle}\, dy\,, \tag{3.1}$$

Remark 3.1. Throughout this section we consider only the case $\mathcal{S}^{\{s\}}(\mathbf{R}^d)$ and remark that the analogous assertions hold when $\mathcal{S}^{\{s\}}(\mathbf{R}^d)$ is replaced by $\mathcal{S}^{(s)}(\mathbf{R}^d)$.

The map $(f,\phi) \mapsto V_\phi f$ from $\mathcal{S}^{\{s\}}(\mathbf{R}^d) \times \mathcal{S}^{\{s\}}(\mathbf{R}^d)$ to $\mathcal{S}^{\{s\}}(\mathbf{R}^{2d})$ extends uniquely to a continuous mapping from $(\mathcal{S}^{\{s\}})'(\mathbf{R}^d) \times (\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$ to $(\mathcal{S}^{\{s\}})'(\mathbf{R}^{2d})$ by duality.

Moreover, for a fixed $\phi \in \mathcal{S}^{\{s\}}(\mathbf{R}^d) \setminus 0$, $s \geq 1$, the following characterization of $\mathcal{S}^{\{s\}}(\mathbf{R}^d)$ holds:

$$f \in \mathcal{S}^{\{s\}}(\mathbf{R}^d) \quad \Longleftrightarrow \quad V_\phi f \in \mathcal{S}^{\{s\}}(\mathbf{R}^{2d}). \tag{3.2}$$

We refer to [13, 28] for the proof and more details on STFT in Gelfand-Shilov spaces.

Now we recall the definition of modulation spaces. Let $s > 1$, $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, $p, q \in [1,\infty]$, and the window $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d) \setminus 0$ be fixed. Then the *modulation space* $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ is the set of all ultra-distributions $f \in (\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$ such that

$$\|f\|_{M^{p,q}_{(\omega)}} = \|f\|_{M^{p,q,\phi}_{(\omega)}} \equiv \|V_\phi f\, \omega\|_{L^{p,q}_1} < \infty. \tag{3.3}$$

Here $\|\cdot\|_{L^{p,q}_1}$ is the norm given by

$$\|F\|_{L^{p,q}_1} \equiv \Big(\int_{\mathbf{R}^d} \Big(\int_{\mathbf{R}^d} |F(x,\xi)|^p\, dx \Big)^{q/p} d\xi \Big)^{1/q},$$

when $F \in L^1_{loc}(\mathbf{R}^{2d})$ (with obvious interpretation when $p = \infty$ or $q = \infty$). Furthermore, the modulation space $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ consists of all $f \in (\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$ such that

$$\|f\|_{W^{p,q}_{(\omega)}} = \|f\|_{W^{p,q,\phi}_{(\omega)}} \equiv \|V_\phi f\, \omega\|_{L^{p,q}_2} < \infty,$$

where $\| \cdot \|_{L_2^{p,q}}$ is the norm given by

$$\|F\|_{L_2^{p,q}} \equiv \Big(\int_{\mathbf{R}^d} \Big(\int_{\mathbf{R}^d} |F(x,\xi)|^q \, d\xi \Big)^{p/q} \, dx \Big)^{1/p},$$

when $F \in L^1_{loc}(\mathbf{R}^{2d})$.

If $\omega = 1$, then the notations $M^{p,q}$ and $W^{p,q}$ are used instead of $M^{p,q}_{(\omega)}$ and $W^{p,q}_{(\omega)}$ respectively. Moreover we set $M^p_{(\omega)} = W^p_{(\omega)} = M^{p,p}_{(\omega)}$ and $M^p = W^p = M^{p,p}$.

We note that $M^{p,q}$ are modulation spaces of classical forms, while $W^{p,q}$ are classical forms of Wiener amalgam spaces. We refer to [4] for the most updated definition of modulation spaces.

If $s > 1$, $p, q \in [1, \infty]$ and $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, then one can show that the spaces $\mathscr{F} L^q_{(\omega)}(\mathbf{R}^d)$, $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ are locally the same, in the sense that

$$\begin{aligned} \mathscr{F} L^q_{(\omega)}(\mathbf{R}^d) \bigcap (\mathcal{E}^{\{s\}})'(\mathbf{R}^d) &= M^{p,q}_{(\omega)}(\mathbf{R}^d) \bigcap (\mathcal{E}^{\{s\}})'(\mathbf{R}^d) \\ &= W^{p,q}_{(\omega)}(\mathbf{R}^d) \bigcap (\mathcal{E}^{\{s\}})'(\mathbf{R}^d). \end{aligned}$$

This follows by similar arguments as in [25] (and replacing the space of polynomially moderated weights $\mathscr{P}(\mathbf{R}^{2d})$ with $\mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$). Later on we extend these properties in the context of wave-front sets and recover the equalities above.

The next proposition concerns topological questions of modulation spaces, and properties of the adjoint of the short-time Fourier transform.

Let $s > 1$, $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, $\phi \in \mathcal{S}^{(s)} \setminus 0$ and $F(x,\xi) \in L^{p,q}_{(\omega)}(\mathbf{R}^{2d})$. Then $V_\phi^* F$ is defined by the formula

$$\langle V_\phi^* F, f \rangle \equiv \langle F, V_\phi f \rangle, \qquad f \in \mathcal{S}^{\{t\}}(\mathbf{R}^d).$$

In what follows we let $L^{p,q}_{(\omega)}(\Omega)$, where $\Omega \subseteq \mathbf{R}^d$, be the set of all $F \in L^1_{loc}(\Omega)$ such that

$$\|F\|_{L^{p,q}_{(\omega)}} \equiv \|F \cdot \omega \chi_\Omega\|_{L^{p,q}},$$

is finite, where χ_Ω is the characteristic function of Ω.

Proposition 3.2. [1] *Let $s > 1$, $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, $p, q \in [1, \infty]$, and $\phi, \phi_1 \in \mathcal{S}^{(s)}(\mathbf{R}^d)$, with $(\phi, \phi_1)_{L^2} \neq 0$. Then the following is true:*

(1) *the operator V_ϕ^* from $\mathcal{S}^{(s)}(\mathbf{R}^{2d})$ to $\mathcal{S}^{(s)}(\mathbf{R}^d)$ extends uniquely to a continuous operator from $L^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ to $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, and*

$$\|V_\phi^* F\|_{M^{p,q}_{(\omega)}} \le C \|V_{\phi_1} \phi\|_{L^1_{(v)}} \|F\|_{L^{p,q}_{(\omega)}}; \tag{3.4}$$

(2) *$M^{p,q}_{(\omega)}(\mathbf{R}^d)$ is a Banach space whose definition is independent on the choice of window $\phi \in \mathcal{S}^{(s)} \setminus 0$;*

(3) *the set of windows can be extended from $\mathcal{S}^{(s)}(\mathbf{R}^d) \setminus 0$ to $M^1_{(v)}(\mathbf{R}^d) \setminus 0$.*

3.2. **Wave-front sets with respect to modulation spaces.** Next we define wave-front sets with respect to modulation spaces and show that they agree with corresponding wave-front sets of Fourier Lebesgue types. More precisely, we prove that [21, Theorem 6.1] holds if the weights of polynomial growth are replaced by more general submultiplicative weights.

Let $s > 1$, $\phi \in \mathcal{S}^{\{s\}}(\mathbf{R}^d) \setminus 0$ ($\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d) \setminus 0$), $\omega \in \mathscr{M}_{\{s\}}$, $\Gamma \subseteq \mathbf{R}^d \setminus 0$ be an open cone and let $p, q \in [1, \infty]$. For any $f \in (\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$ ($f \in (\mathcal{S}^{(s)})'(\mathbf{R}^d)$) we set

$$|f|_{\mathcal{B}(\Gamma)} = |f|_{\mathcal{B}(\phi,\Gamma)} \equiv \|V_\phi f\|_{L^{p,q}_{(\omega)}(\mathbf{R}^d \times \Gamma)} \qquad \text{when} \quad \mathcal{B} = M^{p,q}_{(\omega)} = M^{p,q}_{(\omega)}(\mathbf{R}^d). \tag{3.5}$$

We note that $|f|_{\mathcal{B}(\phi,\Gamma)}$ might attain $+\infty$. Thus we define by $|\cdot|_{\mathcal{B}(\Gamma)}$ a semi-norm on $(\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$ (or on $(\mathcal{S}^{(s)})'(\mathbf{R}^d)$) which might attain the value $+\infty$. If $\Gamma = \mathbf{R}^d \setminus 0$ and $\phi \in \mathcal{S}^{\{s\}}(\mathbf{R}^d)$, then $|f|_{\mathcal{B}(\Gamma)} = \|f\|_{M^{p,q}_{(\omega)}}$.

We also set

$$|f|_{\mathcal{B}(\Gamma)} = |f|_{\mathcal{B}(\phi,\Gamma)} \equiv \Big(\int_{\mathbf{R}^d} \Big(\int_\Gamma |V_\phi f(x,\xi)\omega(x,\xi)|^q \, d\xi \Big)^{p/q} dx \Big)^{1/p} \qquad \text{when} \quad \mathcal{B} = W^{p,q}_{(\omega)} = W^{p,q}_{(\omega)}(\mathbf{R}^d) \tag{3.6}$$

and note that similar properties hold for this semi-norm compared to (3.5).

Let $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, $p, q \in [1, \infty]$, $f \in (\mathcal{S}^{(s)})'(\mathbf{R}^d)$, and let $\mathcal{B} = M^{p,q}_{(\omega)}$ or $\mathcal{B} = W^{p,q}_{(\omega)}$. Then $\Theta_{\mathcal{B}}(f)$, $\Sigma_{\mathcal{B}}(f)$ and the wave-front set $\mathrm{WF}_{\mathcal{B}}(f)$ of f with respect to the modulation space $\mathcal{B}$ are defined in the same way as in Section 1, after replacing the semi-norms of Fourier Lebesgue types in (1.1) with the semi-norms in (3.5) or (3.6) respectively.

We need the following proposition when proving that the wave-front sets of Fourier-Lebesgue and modulation space types are the same. The result is an extension of [1, Proposition 4.2].

Proposition 3.3. *Let $s > 1$. Then the following is true:*

(1) *if $f \in (\mathcal{E}^{\{s\}})'(\mathbf{R}^d)$ and $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d)$, then*

$$|V_\phi f(x,\xi)| \lesssim e^{-h|x|^{1/s}} e^{\varepsilon|\xi|^{1/s}},$$

for every $h > 0$ and every $\varepsilon > 0$;

(2) *if $f \in (\mathcal{D}^{(s)})'(\mathbf{R}^d)$ and in addition $\phi \in \mathcal{D}^{(s)}(\mathbf{R}^d) \setminus 0$, then $f \in (\mathcal{E}^{\{s\}})'(\mathbf{R}^d)$, if and only if $\operatorname{supp} V_\phi f \subseteq K \times \mathbf{R}^d$ for some compact set K and*

$$|V_\phi f(x,\xi)| \lesssim e^{\varepsilon|\xi|^{1/s}},$$

for every $\varepsilon > 0$.

Proof. In order to prove (1) we assume that $f \in (\mathcal{E}^{\{s\}})'(\mathbf{R}^d)$ and $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d)$. Also let $\psi \in \mathcal{D}^{(s)}(\mathbf{R}^d)$ such that $\psi = 1$ in $\operatorname{supp} f$. Then for every $\varepsilon, h > 0$ it holds that

$$|V_\psi \phi(x,\xi)| \lesssim e^{-h|x|^{1/s} - 2\varepsilon|\xi|^{1/s}}.$$

and

$$|\widehat{f}(\xi)| \lesssim e^{\varepsilon|\xi|^{1/s}}.$$

By straight-forward calculations, it follows that

$$\begin{aligned}
|V_\phi f(x,\xi)| &= |(V_\phi(\psi f))(x,\xi)| \lesssim (|V_\psi \phi(x,\cdot)| * |\widehat{f}|)(\xi) \\
&= \int |V_\psi \phi(x,\xi-\eta)||\widehat{f}(\eta)|\, d\eta \lesssim \int e^{-h|x|^{1/s} - 2\varepsilon|\xi-\eta|^{1/s}} e^{\varepsilon|\eta|^{1/s}}\, d\eta \\
&\le e^{-h|x|^{1/s}} \int e^{-2\varepsilon|\eta|^{1/s} + 2\varepsilon|\xi|^{1/s} + \varepsilon|\eta|^{1/s}}\, d\eta
\end{aligned}$$

Now, the assertion follows since both ε and h can be chosen arbitrarily.

Next we prove (2). First assume that $\phi \in \mathcal{D}^{(s)}(\mathbf{R}^d) \setminus 0$ and $f \in (\mathcal{E}^{\{s\}})'(\mathbf{R}^d)$. Since both ϕ and f have compact support, it follows that $\operatorname{supp}(V_\phi f) \subseteq K \times \mathbf{R}^d$. Furthermore,

$$|V_\phi f(x,\xi)| \lesssim e^{\varepsilon(|x|^{1/s} + |\xi|^{1/s})}.$$

Since $V_\phi f(x,\xi)$ has compact support in the x-variable, it follows that

$$|V_\phi f(x,\xi)| \lesssim e^{\varepsilon|\xi|^{1/s}},$$

for every $\varepsilon > 0$.

In order to prove the opposite direction we assume that $\operatorname{supp} V_\phi f \subseteq K \times \mathbf{R}^d$, for some compact set K, and

$$|V_\phi f(x,\xi)| \lesssim e^{\varepsilon|\xi|^{1/s}},$$

for every $\varepsilon > 0$. Then

$$|\widehat{f}(\xi)| = \Big| \int V_\phi f(x,\xi)\, dx \Big| \lesssim e^{\varepsilon|\xi|^{1/s}}, \quad \forall \varepsilon > 0. \tag{3.7}$$

Assume that $\operatorname{supp} \phi \subseteq K$ and choose $\varphi \in \mathcal{D}^{(s)}(\mathbf{R}^d)$ such that $\operatorname{supp} \varphi \cap 2K = \emptyset$. Let $x \notin 2K$, then

$$(f,\varphi) = (V_\phi f, V_\phi \varphi) = 0,$$

which implies that f has compact support. Hence, (3.7) and the fact that $f \in (\mathcal{D}^{(s)})'(\mathbf{R}^d)$ give $f \in (\mathcal{E}^{\{s\}})'(\mathbf{R}^d)$. $\square$

Theorem 3.4. *Let $s > 1$, $p, q \in [1,\infty]$, $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, $\mathcal{B} = \mathscr{F} L^q_{(\omega)}(\mathbf{R}^d)$, and let $\mathcal{C} = M^{p,q}_{(\omega)}(\mathbf{R}^d)$ or $\mathcal{C} = W^{p,q}_{(\omega)}(\mathbf{R}^d)$. If $f \in (\mathcal{D}^{(s)})'(\mathbf{R}^d)$ then*

$$\mathrm{WF}_{\mathcal{B}}(f) = \mathrm{WF}_{\mathcal{C}}(f). \tag{3.8}$$

In particular, $\mathrm{WF}_{\mathcal{C}}(f)$ is independent of p and $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d) \setminus 0$ in (3.5) and (3.6).

In the proof of Theorem 3.4, the main part concerns proving that the wave-front sets of modulation types are independent of the choice of window $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d) \setminus 0$. Note also that the dual pairing between $f \in (\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$ and $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d)$ is well defined.

Proof. We only consider the case $\mathcal{C} = M^{p,q}_{(\omega)}$. The case $\mathcal{C} = W^{p,q}_{(\omega)}$ follows by similar arguments and is left for the reader. We may assume that $f \in (\mathcal{E}^{\{s\}})'(\mathbf{R}^d)$ and that $\omega(x,\xi) = \omega(\xi)$ since the statements only concern local assertions.

In order to prove that $\mathrm{WF}_{\mathcal{C}}(f)$ is independent of $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d) \setminus 0$, we assume that $\phi, \phi_1 \in \mathcal{S}^{(s)}(\mathbf{R}^d) \setminus 0$ and let $|\,\cdot\,|_{\mathcal{C}_1(\Gamma)}$ be the semi-norm in (3.5) after ϕ has been replaced by ϕ_1. Let Γ_1 and Γ_2 be open cones in $\mathbf{R}^d$ such that $\overline{\Gamma_2} \subseteq \Gamma_1$. The asserted independency of ϕ follows if we prove that

$$|f|_{\mathcal{C}(\Gamma_2)} \le C(|f|_{\mathcal{C}_1(\Gamma_1)} + 1), \tag{3.9}$$

for some positive constant C. Let

$$\Omega_1 = \{\, (x,\xi)\,;\, \xi \in \Gamma_1 \,\} \subseteq \mathbf{R}^{2d} \quad \text{and} \quad \Omega_2 = \complement\Omega_1 \subseteq \mathbf{R}^{2d},$$

with characteristic functions χ_1 and χ_2 respectively, and set

$$F_k(x,\xi) = |V_{\phi_1} f(x,\xi)|\omega(\xi)\chi_k(x,\xi), \quad k = 1,2,$$

and $G = |V_\phi \phi_1(x,\xi)|v(\xi)$. Since ω is v-moderate, it follows from [11, Lemma 11.3.3] that

$$|V_\phi f(x,\xi)\omega(x,\xi)| \lesssim \big((F_1 + F_2) * G\big)(x,\xi),$$

which implies that

$$|f|_{\mathcal{C}(\Gamma_2)} \lesssim J_1 + J_2, \tag{3.10}$$

where

$$J_k = \Big(\int_{\Gamma_2} \Big(\int_{\mathbf{R}^d} |(F_k * G)(x,\xi)|^p \, dx \Big)^{q/p} d\xi \Big)^{1/q}, \quad k = 1,2.$$

By Young's inequality

$$J_1 \le \|F_1 * G\|_{L^{p,q}_1} \le \|G\|_{L^1} \|F_1\|_{L^{p,q}_1} = C|f|_{\mathcal{C}_1(\Gamma_1)}, \tag{3.11}$$

where $C = \|G\|_{L^1} = \|V_\phi \phi_1(x,\xi)v(\xi)\|_{L^1} < \infty$, in view of (3.2).

Next we consider J_2. For $\xi \in \Gamma_2$ and $\eta \in \complement\Gamma_1$, it follows from (1.7), Propositon 3.3 and (3.2) that for every $N, l > 0$ and for some $k > 0$ we

may choose $h>0$ such that

$$
\begin{aligned}
|(F_2{*}G)(x,\xi)| &\lesssim \iint_{\mathbf{R}^{2d}} e^{-N|y|^{1/s}} e^{(l+k)|\eta|^{1/s}} e^{-h(|x-y|^{1/s}+|\xi-\eta|^{1/s})} v(\xi-\eta)\, dyd\eta \\
\lesssim \iint_{\mathbf{R}^{2d}} &e^{-N|y|^{1/s}} e^{(l+k)|\eta|^{1/s}} e^{-hc(|x|^{1/s}+|y|^{1/s}+|\xi|^{1/s}+|\eta|^{1/s})/2} e^{k(|\xi|^{1/s}+|\eta|^{1/s})}\, dyd\eta \\
&\lesssim e^{-hc|x|^{1/s}/2} e^{(k-\frac{hc}{2})|\xi|^{1/s}} \iint_{\mathbf{R}^{2d}} e^{-N|y|^{1/s}-hc|y|^{1/s}/2} e^{(l+2k-hc/2)|\eta|^{1/s}}\, dyd\eta, \\
&\lesssim e^{-hc|x|^{1/s}/2} e^{(k-hc/2)|\xi|^{1/s}} < \infty.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
J_2 &= \Big(\int_{\Gamma_2} \Big(\int_{\mathbf{R}^d} |(F_2 * G)(x,\xi)|^p \, dx \Big)^{q/p} d\xi \Big)^{1/q} \\
&\lesssim \Big(\int_{\Gamma_2} \Big(\int_{\mathbf{R}^d} e^{-hc|x|^{1/s}/2} e^{(k-hc/2)|\xi|^{1/s}} \Big)^p dx \Big)^{q/p} d\xi \Big)^{1/q} < \infty.
\end{aligned}
$$

This proves that (3.9), and hence $\mathrm{WF}_{\mathcal{C}}(f)$ is independent of $\phi \in \mathcal{S}^{(s)}(\mathbf{R}^d)\setminus 0$.

In order to prove (3.8) we assume from now on that ϕ in (3.5) is real-valued and has compact support. Let $p_0 \in [1,\infty]$ be such that $p_0 \le p$ and set $\mathcal{C}_0 = M^{p_0,q}_{(\omega)}$. The result follows if we prove

$$
\Theta_{\mathcal{C}_0}(f) \subseteq \Theta_{\mathcal{B}}(f) \subseteq \Theta_{\mathcal{C}}(f) \quad \text{when } p_0 = 1,\ p = \infty, \tag{3.12}
$$

and

$$
\Theta_{\mathcal{C}}(f) \subseteq \Theta_{\mathcal{C}_0}(f). \tag{3.13}
$$

The proof of the first inclusion in (3.12) follows from the estimates

$$
\begin{aligned}
|f|_{\mathcal{B}(\Gamma)} &\lesssim \Big(\int_\Gamma |\widehat{f}(\xi)\omega(\xi)|^q \, d\xi \Big)^{1/q} \\
&\lesssim \Big(\int_\Gamma |\mathscr{F}\Big(f \int_{\mathbf{R}^d} \phi(\cdot - x)\, dx \Big)(\xi)\omega(\xi)|^q \, d\xi \Big)^{1/q} \\
&\lesssim \Big(\int_\Gamma \Big(\int_{\mathbf{R}^d} |\mathscr{F}(f\phi(\cdot - x))(\xi)\omega(\xi)| \, dx \Big)^q d\xi \Big)^{1/q} \\
&= \Big(\int_\Gamma \Big(\int_{\mathbf{R}^d} |V_\phi f(x,\xi)\omega(\xi)| \, dx \Big)^q d\xi \Big)^{1/q} = C|f|_{\mathcal{C}_0(\Gamma)},
\end{aligned}
$$

for some positive constant C.

Next we prove the second inclusion in (3.12). We have

$$
\begin{aligned}
|f|_{\mathcal C(\Gamma_2)} &= \Big(\int_{\Gamma_2} \sup_{x\in \mathbf R^d} |V_\phi f(x,\xi)\omega (x,\xi)|^q\, d\xi \Big)^{1/q}
\\
&\lesssim \Big(\int_{\Gamma_2} \sup_{x\in \mathbf R^d} |(|\widehat f| * |\mathscr F(\phi(\cdot - x))|)(\xi)\omega (\xi)|^q\, d\xi \Big)^{1/q}
\\
&\lesssim \Big(\int_{\Gamma_2} |(|\widehat f| * |\widehat \phi|)(\xi)\omega (\xi)|^q\, d\xi \Big)^{1/q}
\\
&\lesssim \Big(\int_{\Gamma_2} ((|\widehat f\cdot \omega| * |\widehat \phi \cdot v|)(\xi))^q\, d\xi \Big)^{1/q},
\end{aligned}
$$

where $\phi \in \mathcal D^{(s)}(X)$ is chosen such that $\phi =1$ in $\operatorname{supp} f$. The second inclusion in (3.12) now follows by straight-forward computations, using similar arguments as in the proof of (1.5). The details are left for the reader.

It remains to prove (3.13). Let $K\subseteq \mathbf R^d$ be compact and chosen such that $V_\phi f(x,\xi)=0$ outside K, and let $p_1\in [1,\infty]$ be chosen such that $1/p_1+1/p_0=1+1/p$. By Hölder's inequality we get

$$
\begin{aligned}
|f|_{\mathcal C_0(\Gamma)} &= \Big(\int_{\Gamma} \Big(\int_{\mathbf R^d} |V_\phi f(x,\xi)\omega (x,\xi)|^{p_0}\, dx \Big)^{q/p_0}\, d\xi \Big)^{1/q}
\\
&\le C_K \Big(\int_{\Gamma} \Big(\int_{\mathbf R^d} |V_\phi f(x,\xi)\omega (x,\xi)|^{p}\, dx \Big)^{q/p}\, d\xi \Big)^{1/q} = C_K|f|_{\mathcal C(\Gamma)}.
\end{aligned}
$$

This gives (3.13), and the proof is complete.

□

Corollary 3.5. *Let $s>1$, $p,q\in [1,\infty]$, and $\omega \in \mathscr M_{\{s\}}(\mathbf R^{2d})$. If $f\in (\mathcal S^{\{s\}})'(\mathbf R^d)$ is compactly supported, then*

$$
f\in \mathcal B \quad \Longleftrightarrow \quad \mathrm{WF}_{\mathcal B}(f)=\emptyset ,
$$

where $\mathcal B$ is equal to $\mathscr FL^q_{(\omega)}$, $M^{p,q}_{(\omega)}$ or $W^{p,q}_{(\omega)}$.

In particular, we recover Corollary 6.2 in [21], Theorem 2.1 and Remark 4.6 in [25].

4. Discrete semi-norms in Fourier Lebesgue spaces

In this section we introduce discrete analogues of the semi-norms in (1.1) and (3.5), and show that these semi-norms are finite if and only if the corresponding non-discrete semi-norms are finite. The techniques used here are similar to those in [16].

Assume that $q \in [1,\infty]$, $s>1$, $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^d)$, $\mathcal{B} = \mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$, and $H \subset \mathbf{R}^d$ is a discrete set. Then we set

$$|f|^{(D)}_{\mathcal{B}(H)} \equiv \left(\sum_{\xi_l \in H} |\widehat{f}(\xi_l)\omega(\xi_l)|^q \right)^{1/q}, \qquad \widehat{f} \in C(\mathbf{R}^d) \cap (\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$$

with obvious modifications when $q=\infty$. As in the continuous case, we may allow weight functions in $\mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, i. e. $\omega = \omega(x,\xi)$. However, again we note that the condition

$$|f|^{(D)}_{\mathcal{B}(H)} < \infty$$

is independent of $x \in \mathbf{R}^d$.

By a lattice Λ we mean the set

$$\Lambda = \{a_1 e_1 + \cdots + a_d e_d;\, a_1,\dots,a_d \in \mathbf{Z}\},$$

where $e_1,\dots,e_d$ is a basis in $\mathbf{R}^d$.

The following Lemma was proved for distributions, cf. [16, 26, 27].

Lemma 4.1. *Let $s>1$, $f \in (\mathcal{E}^{\{s\}})'(\mathbf{R}^d)$, Γ and Γ_0 be open cones in $\mathbf{R}^d \setminus 0$ such that $\overline{\Gamma_0} \subseteq \Gamma$, $q \in [1,\infty]$, and let $\Lambda \subseteq \mathbf{R}^d$ be a lattice. If $|f|_{\mathcal{B}(\Gamma)}$ is finite, then $|f|^{(D)}_{\mathcal{B}(\Gamma_0 \cap \Lambda)}$ is finite.*

Proof. We only prove the result for $q<\infty$, leaving the small modifications in the case $q=\infty$ for the reader. Assume that $|f|_{\mathcal{B}(\Gamma)}<\infty$, and let $H = \Gamma_0 \cap \Lambda$. Also let $\varphi \in \mathcal{D}^{(s)}(\mathbf{R}^d)$ be such that $\varphi = 1$ in $\operatorname{supp} f$. Then

$$\begin{aligned}
(|f|^{(D)}_{\mathcal{B}(\Gamma_0\cap\Lambda)})^q &= \sum_{\xi_l \in H} |\mathscr{F}(\varphi f)(\xi_l)\omega(\xi_l)|^q \\
&= (2\pi)^{-qd/2} \sum_{\xi_l \in H} \left| \int \widehat{\varphi}(\xi_l - \eta)\widehat{f}(\eta)\omega(\xi_l)\, d\eta \right|^q \lesssim (S_1 + S_2),
\end{aligned}$$

where

$$S_1 = \sum_{\xi_l \in H} \left(\int_{\Gamma} \psi(\xi_l - \eta) F(\eta)\, d\eta \right)^q,$$

$$S_2 = \sum_{\xi_l \in H} \left(\int_{\complement\Gamma} \psi(\xi_l - \eta) F(\eta)\, d\eta \right)^q.$$

Here we set $F(\xi) = |\widehat{f}(\xi)\omega(\xi)|$ and $\psi(\xi) = |\widehat{\varphi}(\xi)v(\xi)|$ as in the proof of Theorem 1.2.

We need to estimate S_1 and S_2. Let c be chosen such that ω is moderate with respect to $v = e^{c|\cdot|^{1/s}}$. By Hölder's inequality we get

$$
\begin{aligned}
S_1 &\le C' \sum_{\xi_l \in H} \Big(\int_\Gamma \psi(\xi_l - \eta) F(\eta)\, d\eta \Big)^q \\
&= C' \sum_{\xi_l \in H} \Big(\int_\Gamma \psi(\xi_l - \eta)^{1/q'} (\psi(\xi_l - \eta)^{1/q} F(\eta))\, d\eta \Big)^q \\
&\le C' \|F\|_{L^1}^{q/q'} \sum_{\xi_l \in H} \int_\Gamma \psi(\xi_l - \eta) F(\eta))^q\, d\eta \\
&\le C'' \int_\Gamma F(\eta)^q\, d\eta = C'' |f|^q_{\mathcal B(\Gamma)},
\end{aligned}
$$

where

$$
C'' = C' \|\psi\|_{L^1}^{q/q'} \sup_{\eta \in \mathbf R^d} \sum_{\xi_l \in H} \psi(\xi_l - \eta)
$$

is finite by (1.6). This proves that S_1 is finite.

It remains to prove that S_2 is finite. We observe that

$$
|\xi_l - \eta|^{1/s} \ge c \max(|\xi_l|^{1/s}, |\eta|^{1/s}) \ge \frac{c}{2}(|\xi_l|^{1/s} + |\eta|^{1/s}) \qquad \text{when} \quad \xi_l \in H \quad \text{and} \quad \eta \in \complement\Gamma.
$$

Since $f \in (\mathcal E^{\{s\}})'(\mathbf R^d)$ it follows that

$$
|F| \lesssim e^{c|\cdot|^{1/s}}
$$

for every positive constant c. Furthermore, since $\varphi \in \mathcal D^{(s)}(\mathbf R^d)$, it follows that for each $N \ge 0$, there is a constant C_N such that $\psi \le C_N e^{-N|\cdot|^{1/s}}$. This gives

$$
\begin{aligned}
S_2 &\lesssim \sum_{\xi_l \in H} \Big(\int_{\complement\Gamma} e^{-N|\xi_l - \eta|^{1/s}} e^{k|\eta|^{1/s}}\, d\eta \Big)^q \\
&\lesssim \sum_{\xi_l \in H} e^{-qNc/2|\xi_l|^{1/s}} \Big(\int e^{-(Nc/2-k)|\eta|^{1/s}}\, d\eta \Big)^q ,
\end{aligned}
$$

where we have used the fact that ω is v-moderate. The result now follows, since the right-hand side is finite when $N > 2k/c$. The proof is complete. $\square$

Next we prove a converse of Lemma 4.1, in the case when the lattice Λ is dense enough. Let $e_1, \dots, e_d$ in $\mathbf R^d$ be a basis for Λ, i. e. for some $x_0 \in \Lambda$ we have

$$
\Lambda = \{x_0 + t_1 e_1 + \dots + t_d e_d;\ t_1, \dots, t_d \in \mathbf Z\}.
$$

A parallelepiped D, spanned by $e_1, \dots, e_d$ for Λ and with corners in Λ, is called a Λ-parallelepiped. This means that for some $x_0 \in \Lambda$ and some basis $e_1, \dots, e_d$ for Λ we have

$$D = \{x_0 + t_1 e_1 + \dots + t_d e_d;\ t_1, \dots, t_d \in [0,1]\}.$$

We let $\mathcal{A}(\Lambda)$ be the set of all Λ-parallelepipeds. For future references we note that if $D_1, D_2 \in \mathcal{A}(\Lambda)$, then their volumes $|D_1|$ and $|D_2|$ agree, and for convenience we let $\|\Lambda\|$ denote the common value, i. e.

$$\|\Lambda\| = |D_1| = |D_2|.$$

Let Λ_1 and Λ_2 be lattices in $\mathbf{R}^d$ with bases $e_1, \dots, e_d$ and $\varepsilon_1, \dots, \varepsilon_d$ respectively. Then the pair (Λ_1, Λ_2) is called *admissible lattice pair*, if for some $0 < c \le 2\pi$ we have $\langle e_j, \varepsilon_j \rangle = c$ and $\langle e_j, \varepsilon_k \rangle = 0$ when $j \neq k$. If in addition $c < 2\pi$, then (Λ_1, Λ_2) is called a *strongly admissible lattice pair*. If instead $c = 2\pi$, then the pair (Λ_1, Λ_2) is called a *weakly admissible lattice pair*.

Lemma 4.2. *Let $s > 1$, (Λ_1, Λ_2) be an admissible lattice pair, $D_1 \in \mathcal{A}(\Lambda_1)$ (be open), and let $f \in (\mathcal{E}^{\{s\}})'(\mathbf{R}^d)$ be such that an open neighbourhood of its support is contained in D_1. Also let Γ and Γ_0 be open cones in $\mathbf{R}^d$ such that $\overline{\Gamma_0} \subseteq \Gamma$. If $|f|^{(D)}_{\mathcal{B}(\Gamma \cap \Lambda_2)}$ is finite, then $|f|_{\mathcal{B}(\Gamma_0)}$ is finite.*

Proof. Since D_1 contains an open neighbourhood of the support of f, we may modify Λ_1 (and therefore D_1) such that the lattice pair (Λ_1, Λ_2) is strongly admissible, and such that the hypothesis still holds. From now on we therefore assume that (Λ_1, Λ_2) is strongly admissible.

We use similar arguments as in the proof of Lemma 4.1. Again we prove the result only for $q < \infty$. The small modifications to the case $q = \infty$ are left for the reader.

Assume that $|f|^{(D)}_{\mathcal{B}(\Gamma \cap \Lambda_2)} < \infty$, and let $\varphi \in \mathcal{D}^{(s)}(D_1^\circ)$ be equal to one in the support of f, where D_1° denotes the interior of the set D_1.

By expanding $f = \varphi f$ into a Fourier series on D_1 we get

$$\widehat{f}(\xi) = C \sum_{\xi_l \in \Lambda_2} \widehat{\varphi}(\xi - \xi_l) \widehat{f}(\xi_l),$$

where the positive constant C only depends on Λ_2. We have

$$\begin{aligned}(|f|_{\mathcal{B}(\Gamma_0)})^q &= \int_{\Gamma_0} |\widehat{f}(\xi)\omega(\xi)|^q \, d\xi \\ &= C^q \int_{\Gamma_0} \left| \sum_{\xi_l \in \Lambda_2} \widehat{\varphi}(\xi - \xi_l) \widehat{f}(\xi_l) \omega(\xi) \right|^q d\xi \le C^q (S_1 + S_2),\end{aligned}$$

where

$$S_1 = \int_{\Gamma_0} \left| \sum_{\xi_l \in H_1} \widehat{\varphi}(\xi - \xi_l) \widehat{f}(\xi_l) \omega(\xi) \right|^q \, d\xi,$$

$$S_2 = \int_{\Gamma_0} \left| \sum_{\xi_l \in H_2} \widehat{\varphi}(\xi - \xi_l) \widehat{f}(\xi_l) \omega(\xi) \right|^q \, d\xi,$$

$H_1 = \Gamma \cap \Lambda_2$ and $H_2 = \complement\Gamma \cap \Lambda_2$.

We have to estimate S_1 and S_2. Let ω be moderate with respect to the weight $v(\cdot) = e^{k|\cdot|^{1/s}}$. By Minkowski's inequality we get

$$\begin{aligned} S_1 &\le C \int_{\Gamma_0} \left(\sum_{\xi_l \in H_1} |\widehat{\varphi}(\xi - \xi_l) v(\xi - \xi_l)| |\widehat{f}(\xi_l) \omega(\xi_l)| \right)^q \, d\xi \\ &\le C' \int_{\Gamma_0} \left(\sum_{\xi_l \in H_1} |\widehat{\varphi}(\xi - \xi_l) v(\xi - \xi_l)| |\widehat{f}(\xi_l) \omega(\xi_l)|^q \right) \, d\xi \\ &\le C'' \sum_{\xi_l \in H_1} |\widehat{f}(\xi_l) \omega(\xi_l)|^q, \end{aligned}$$

where

$$C' = C \sup_{\xi} \|\widehat{\varphi}(\xi - \xi_l) v(\xi - \xi_l)\|_{l^1(\Lambda_2)}^{q/q'} < \infty, \quad \text{and} \quad C'' = C' \|\varphi\|_{\mathscr{F}L^1_{(v)}} < \infty.$$

This proves that S_1 is finite when $|f|^{(D)}_{\mathcal{B}(\Gamma \cap \Lambda_2)} < \infty$.

It remains to prove that S_2 is finite. We recall that

$$|\xi - \xi_l|^{1/s} \ge c \max(|\xi|^{1/s}, |\xi_l|^{1/s}) \ge c(|\xi|^{1/s} + |\xi_l|^{1/s})/2 \qquad \text{when} \quad \xi \in \Gamma_0 \text{ and } \xi_l \in H_2,$$

and use the same arguments as in the proof of Lemma 4.1 to obtain

$$\begin{aligned} S_2 &\lesssim \int_{\Gamma_0} \left(\sum_{\xi_l \in H_2} e^{-N|\xi - \xi_l|^{1/s}} e^{k|\xi_l|^{1/s}} \right)^q \, d\xi \\ &\lesssim \int_{\Gamma_0} e^{-qNc/2|\xi|^{1/s}} \left(\sum_{\xi_l \in H_2} e^{-(Nc/2 - k)|\xi_l|^{1/s}} \right)^q \, d\xi. \end{aligned}$$

The result now follows, since the right-hand side is finite when $N > 2k/c$. The proof is complete. □

Corollary 4.3. *Let $s > 1$, (Λ_1, Λ_2) be an admissible lattice pair, $D_1 \in \mathcal{A}(\Lambda_1)$, and let $f \in (\mathcal{E}^{\{s\}})'(\mathbf{R}^d)$ be such that an open neighbourhood of its support is contained in D_1. Also let Γ and Γ_0 be open cones in $\mathbf{R}^d$*

such that $\overline{\Gamma_0} \subseteq \Gamma$. If $|f|^{(D)}_{\mathcal{B}(\Gamma\cap\Lambda_2)}$ is finite, then $|\varphi f|^{(D)}_{\mathcal{B}(\Gamma_0\cap\Lambda_2)}$ is finite for every $\varphi \in \mathcal{D}^{(s)}(X)$.

For the proof we recall that $|\varphi f|_{\mathcal{B}(\Gamma_0)}$ is finite when $f \in (\mathcal{E}^{\{s\}})'(\mathbf{R}^d)$, $\varphi \in \mathcal{D}^{\{s\}}(X)$, and $|f|_{\mathcal{B}(\Gamma)}$ is finite. This follows from the proof of Theorem 1.2.

Proof. Let Γ_1, Γ_2 be open cones such that $\overline{\Gamma_j} \subseteq \Gamma_{j+1}$ for $j = 0, 1$, $\overline{\Gamma_2} \subseteq \Gamma$, and assume that $|f|^{(D)}_{\mathcal{B}(\Gamma\cap\Lambda_2)} < \infty$. Then Lemma 4.2 shows that $|f|_{\mathcal{B}(\Gamma_2)}$ is finite. Hence, Theorem 1.2 implies that $|\varphi f|_{\mathcal{B}(\Gamma_1)} < \infty$. This gives $|\varphi f|^{(D)}_{\mathcal{B}(\Gamma_0\cap\Lambda_2)} < \infty$, in view of Lemma 4.1. The proof is complete. □

5. Gabor pairs

In this section we introduce in Definition 5.1 the notion of Gabor pairs. We refer to [16] for an explanation that conditions in Definition 5.1 are quite general.

By Definition 5.1 it follows that our analysis can be applied to the most general classes of non-quasianalytic ultradistributions, and it also points out the role of Beurling-Domar weights in definitions of $\mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$ and $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, cf. [3, 12, 13]. On the other hand, a larger class of quasianalytic ultradistributions can not be treated by the technique given here, since the corresponding test function spaces do not contain smooth functions of compact support.

Assume that $e_1, \dots, e_d$ is a basis for the lattice Λ_1, and that (Λ_1, Λ_2) is a weakly admissible lattice pair. If $f \in L^2_{\mathrm{loc}}$ is periodic with respect to Λ_1, and D is the parallelepiped, spanned by $\{e_1, \dots, e_d\}$, then we may make Fourier expansion of f as

$$f(x) = \sum_{\xi_l \in \Gamma_2} c_l e^{i\langle x, \xi_l\rangle}, \qquad x \in \mathbf{R}^d \tag{5.1}$$

(with convergence in L^2_{loc}), where the coefficients c_l are given by

$$c_l = \int_\Delta f(y) e^{-i\langle y, \xi_l\rangle}\, dy. \tag{5.2}$$

Here and in what follows we let

$$y = y_1 e_1 + \dots + y_d e_d, \qquad dy = dy_1 \cdots dy_d \qquad \text{and} \qquad \Delta = [0,1]^d. \tag{5.3}$$

For non-periodic functions and distributions we instead make Gabor expansions. Because of the support properties of the involved Gabor atoms and their duals, we are usually forced to change the assumption on the involved lattice pairs. More precisely, instead of assuming that (Λ_1, Λ_2) should be a weakly admissible lattice pair, we assume from now on that (Λ_1, Λ_2) is a strongly admissible lattice pair, with $\Lambda_1 = \{x_j\}_{j\in J}$

and $\Lambda_2 = \{\xi_l\}_{l\in J}$. Also let $s>1$ and

$$\begin{aligned}\phi,\psi \in \mathcal{D}^{(s)}(\mathbf{R}^d), \qquad \phi_{j,l}(x) =&\phi(x-x_j)e^{i\langle x,\xi_l\rangle}\\ \text{and} \quad \psi_{j,l}(x) = \psi(x-x_j)&e^{i\langle x,\xi_l\rangle}\end{aligned} \tag{5.4}$$

be such that $\{\phi_{j,l}\}_{j,l\in J}$ and $\{\psi_{j,l}\}_{j,l\in J}$ are dual Gabor frames (see [7,11] for the definition and basic properties of Gabor frames and their duals). If $f \in (\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$ then

$$f = \sum_{j,l\in J} c_{j,l}\phi_{j,l}, \tag{5.5}$$

where

$$c_{j,l} = C_{\phi,\psi}(f,\psi_{j,l})_{L^2(\mathbf{R}^d)} \tag{5.6}$$

and the constant $C_{\phi,\psi}$ depends on the frames only.

Note that the convergence is in $(\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$ due to Proposition 3.3.

Definition 5.1. Assume that $\varepsilon \in (0,1]$, $\{x_j\}_{j\in J} = \Lambda_1 \subseteq \mathbf{R}^d$ and $\{\xi_l\}_{l\in J} = \Lambda_2 \subseteq \mathbf{R}^d$ are lattices and let $\Lambda_1(\varepsilon) = \varepsilon\Lambda_1$. Also let $\phi,\psi \in C_0^\infty(\mathbf{R}^d)$ be non-negative, and set

$$\begin{aligned}\phi^\varepsilon &= \phi(\cdot/\varepsilon), & \psi^\varepsilon &= \psi(\cdot/\varepsilon),\\ \phi^\varepsilon_{j,l} &= \phi^\varepsilon(\cdot-\varepsilon x_j)e^{i\langle\cdot,\xi_l\rangle}, & \psi^\varepsilon_{j,l} &= \psi^\varepsilon(\cdot-\varepsilon x_j)e^{i\langle\cdot,\xi_l\rangle}\end{aligned} \tag{5.7}$$

when $\varepsilon x_j \in \Lambda_1(\varepsilon)$ (i.e. $x_j \in \Lambda_1$) and $\xi_l \in \Lambda_2$. Then the pair

$$(\{\phi_{j,l}\}_{j,l\in J}, \{\psi_{j,l}\}_{j,l\in J}) \tag{5.8}$$

is called a Gabor pair with respect to the lattices Λ_1 and Λ_2 if for each $\varepsilon \in (0,1]$, the sets $\{\phi^\varepsilon_{j,l}\}_{j,l\in J}$ and $\{\psi^\varepsilon_{j,l}\}_{j,l\in J}$ are dual Gabor frames.

By Definition 5.1 and Chapters 5-13 in [11] it follows that if $f \in (\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$ and if $(\{\phi_{j,l}\}_{j,l\in J}, \{\psi_{j,l}\}_{j,l\in J})$ is a Gabor pair, then

$$f = \sum_{j,l\in J} c_{j,l}(\varepsilon)\phi^\varepsilon_{j,l} \tag{5.5$''$}$$

in $(\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$, for every $\varepsilon \in (0,1]$, where

$$c_{j,l}(\varepsilon) = (f,\psi^\varepsilon_{j,l}). \tag{5.6$''$}$$

Here $(\cdot,\cdot)$ denotes the unique extension of the L^2-form on $\mathcal{S}^{\{s\}}(\mathbf{R}^d)\times\mathcal{S}^{\{s\}}(\mathbf{R}^d)$ into $(\mathcal{S}^{\{s\}})'(\mathbf{R}^d)\times(\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$.

We remark that if the pair in (5.8) is a Gabor pair, then it follows from the investigations in [11] that the lattice pair (Λ_1,Λ_2) in Definition 5.1 is strongly admissible.

The following proposition explains that any pair of dual Gabor frames satisfying a mild additional condition, generates a Gabor pair.

Proposition 5.2. [16] *Let* $\phi, \psi \in C_0^\infty(\mathbf{R}^d)$ *be non-negative functions and let* $\phi_{j,l}$ *and* $\psi_{j,l}$ *be given by* (5.4)*. Also, let* Λ_1 *and* Λ_2 *be the same as in Definition 5.1. If* $\{\phi_{j,l}\}_{j,l\in J}$ *and* $\{\psi_{j,l}\}_{j,l\in J}$ *are dual Gabor frames such that*

$$\sum_{x_j\in\Lambda_1} \phi(\cdot - x_j)\psi(\cdot - x_j) = \|\Lambda_1\|^{-1}, \tag{5.9}$$

holds, then (5.8) *is a Gabor pair.*

Remark 5.3. If $\phi = \psi$, then (5.9) describes the tight frame property of the corresponding Gabor frame, cf. [11, Theorem 6.4.1].

Remark 5.4. Let $p, q \in [1,\infty]$, $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$, and $f \in (\mathcal{E}^{\{s\}})'(\mathbf{R}^d)$. If $(\{\phi_{j,l}\}_{j,l\in J}, \{\psi_{j,l}\}_{j,l\in J})$ is a Gabor pair such that (5.5) and (5.6) hold, then it follows that $f \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$ if and only if

$$\|f\|_{[\varepsilon]} \equiv \Big(\sum_{l\in J} \Big(\sum_{j\in J} |c_{j,l}(\varepsilon)\omega(\varepsilon x_j, \xi_j)|^p \Big)^{q/p} \Big)^{1/q}$$

is finite for every $\varepsilon \in (0,1]$. Furthermore, for every $\varepsilon \in (0,1]$, the norm $f \mapsto \|f\|_{[\varepsilon]}$ is equivalent to the modulation space norm (1.3) (cf. [3,5,6,11].)

6. Discrete versions of wave-front sets

In this section we define discrete wave-front sets with respect to Fourier Lebesgue and modulation spaces, and prove that they agree with the corresponding wave-front sets of continuous types. In the first part we consider discrete versions with respect to Fourier Lebesgue and modulation spaces, and show that they agree with each other, and also with the corresponding continuous ones. In the second part we consider more general situations, where we discuss similar questions for sequences of spaces. In such a way we are able to characterize Hörmander's wave-front sets with our discrete approach.

6.1. **Discrete versions of wave-front sets with respect to Fourier Lebesgue and modulation spaces.** We start with two definitions.

Definition 6.1. Let $s > 1$, $q \in [1,\infty]$, $f \in (\mathcal{S}^{\{s\}})'(\mathbf{R}^d)$, and let (Λ_1, Λ_2) be a strongly admissible lattice pair in $\mathbf{R}^d$ such that $x_0 \notin \Lambda_1$. Moreover, let $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^d)$ and $\mathcal{B} = \mathscr{F}L^q_{(\omega)}(\mathbf{R}^d)$. Then the discrete wave-front set $\mathrm{DF}_{\mathcal{B}}(f)$ consists of all $(x_0, \xi_0) \in X \times (\mathbf{R}^d \setminus 0)$, $X \subseteq \mathbf{R}^d$ is open such that for each $\varphi \in \mathcal{D}^{(s)}(X)$ with $\varphi(x_0) \neq 0$ and each open conical neighbourhood Γ of ξ_0, it holds

$$|\varphi f|^{(D)}_{\mathcal{B}(\Gamma)} = \infty.$$

For the definition of discrete wave-front sets of modulation spaces, we consider Gabor pairs $(\{\phi_{j,l}\}_{j,l\in J}, \{\psi_{j,l}\}_{j,l\in J})$, and let

$$J_{x_0}(\varepsilon) = J_{x_0}(\varepsilon, \phi, \psi) = J_{x_0}(\varepsilon, \phi, \psi, \Lambda_1)$$

be the set of all $j \in J$ such that

$$x_0 \in \operatorname{supp} \phi_{j,l}^{\varepsilon} \quad \text{or} \quad x_0 \in \operatorname{supp} \psi_{j,l}^{\varepsilon}.$$

Definition 6.2. Let $s > 1$, $p, q \in [1, \infty]$, $f \in (\mathcal{S}^{\{s\}})'(X)$, and let $\phi, \psi \in \mathcal{D}^{(s)}(\mathbf{R}^d)$ be non-negative such that $(\{\phi_{j,l}\}_{j,l \in J}, \{\psi_{j,l}\}_{j,l \in J})$ is a Gabor pair with respect to the lattices Λ_1 and Λ_2 in $\mathbf{R}^d$. Moreover, let $\omega \in \mathscr{M}_{\{s\}}(\mathbf{R}^{2d})$ and $\mathcal{C} = M^{p,q}_{(\omega)}(\mathbf{R}^d)$. Then the discrete wave-front set $\mathrm{DF}_{\mathcal{C}}(f)$ consists of all $(x_0, \xi_0) \in X \times (\mathbf{R}^d \setminus 0)$, $X \subseteq \mathbf{R}^d$ is open, such that for each $\varepsilon \in (0, 1]$ and each open conical neighbourhood Γ of ξ_0, it holds

$$\Big(\sum_{\xi_l \in \Gamma \cap \Lambda_2} \Big(\sum_{j \in J_{x_0}(\varepsilon)} |c_{j,l}(\varepsilon) \omega(\xi_l)|^p \Big)^{q/p} \Big)^{1/q} = \infty,$$

where

$$f = \sum_{j,l \in J} c_{j,l(\varepsilon)} \phi_{j,l}^{\varepsilon}, \quad c_{j,l}(\varepsilon) = C_{\phi,\psi} (f, \psi_{j,l}^{\varepsilon})_{L^2(\mathbf{R}^d)}$$

and the constant $C_{\phi,\psi}$ depends on ϕ and ψ only.

Roughly speaking, $(x_0, \xi_0) \in \mathrm{DF}_{\mathcal{C}}(f)$ means that f is not locally in $\mathcal{C}$, in the direction ξ_0. The following result shows that our wave-front sets coincide.

Theorem 6.3. *Let $s > 1$, $X \subseteq \mathbf{R}^d$ be open and let $f \in (\mathcal{D}^{\{s\}})'(X)$. Then*

$$\mathrm{WF}_{\mathcal{B}}(f) = \mathrm{WF}_{\mathcal{C}}(f) = \mathrm{DF}_{\mathcal{B}}(f) = \mathrm{DF}_{\mathcal{C}}(f). \tag{6.1}$$

Proof. By Theorem 3.4 and Lemmas 4.1 and 4.2, it follows that the first two equalities in (6.1) hold. The result therefore follows if we prove that $\mathrm{DF}_{\mathcal{B}}(f) = \mathrm{DF}_{\mathcal{C}}(f)$.

First assume that $(x_0, \xi_0) \notin \mathrm{DF}_{\mathcal{B}}(f)$, and choose $\varphi \in \mathcal{D}^{(s)}(X)$, where $1 < t < s$, an open neighbourhood $X_0 \subset \overline{X_0} \subset X$ of x_0 and conical neighbourhoods Γ, Γ_0 of ξ_0 such that

- $\overline{\Gamma_0} \subseteq \Gamma$, $\quad \varphi(x) = 1 \quad$ when $\quad x \in X_0$,
- $|\varphi\, f|^{(D)}_{\mathcal{B}(H)} < \infty$, $\quad$ when $\quad H = \Lambda_2 \cap \Gamma$.

Now let $(\{\phi_{j,l}\}_{j,l \in J}, \{\psi_{j,l}\}_{j,l \in J})$ be a Gabor pair and choose $\varepsilon \in (0, 1]$ such that $\operatorname{supp} \phi_{j,l}^{\varepsilon}$ and $\operatorname{supp} \psi_{j,l}^{\varepsilon}$ are contained in X_0 when $x_0 \in \operatorname{supp} \phi_{j,l}^{\varepsilon}$ and $x_0 \in \operatorname{supp} \psi_{j,l}^{\varepsilon}$. Since

$$c_{j,l}(\varepsilon) = C (f, \psi_{j,l}^{\varepsilon})_{L^2(\mathbf{R}^d)} = \mathscr{F}(f\, \psi(\,\cdot\,/\varepsilon - x_j))(\xi_l),$$

it follows from these support properties that if $H_0 = \Lambda_2 \cap \Gamma_0$, then

$$\Big(\sum_{\xi_l \in H_0} |\mathscr{F}(f\, \psi(\,\cdot\,/\varepsilon - x_j))(\xi_l) \omega(\xi_l)|^q \Big)^{1/q}$$

$$= |f\, \psi(\,\cdot\,/\varepsilon - x_j)|^{(D)}_{\mathcal{B}(H_0)} = |f\, \varphi \psi(\,\cdot\,/\varepsilon - x_j)|^{(D)}_{\mathcal{B}(H_0)}, \tag{6.2}$$

when $j \in J_{x_0}(\varepsilon)$. Hence, by combining Corollary 4.3 with the facts that $J_{x_0}(\varepsilon)$ is finite and $|\varphi\, f|^{(D)}_{\mathcal{B}(H)} < \infty$, it follows that the expressions in (6.2) are finite and

$$\Big(\sum_{\xi_l \in H_0} \Big(\sum_{j \in J_{x_0}(\varepsilon)} |\mathscr{F}(f\, \psi(\,\cdot\,/\varepsilon - x_j))(\xi_l)\omega(\xi_l)|^p \Big)^{q/p} \Big)^{1/q} < \infty.$$

This implies that $(x_0, \xi_0) \notin \mathrm{DF}_{\mathcal{C}}(f)$, and we have proved that $\mathrm{DF}_{\mathcal{C}}(f) \subseteq \mathrm{DF}_{\mathcal{B}}(f)$.

In order to prove the opposite inclusion we assume that $(x_0, \xi_0) \notin \mathrm{DF}_{\mathcal{C}}(f)$, and we choose $\varepsilon \in (0,1]$, Gabor pair $(\{\phi_{j,l}\}_{j,l \in J}, \{\psi_{j,l}\}_{j,l \in J})$ and conical neighbourhoods Γ, Γ_0 of ξ_0 such that $\overline{\Gamma_0} \subseteq \Gamma$ and

$$\Big(\sum_{\xi_l \in H} \Big(\sum_{j \in J_{x_0}(\varepsilon)} |\mathscr{F}(f\, \psi(\,\cdot\,/\varepsilon - x_j))(\xi_l)\omega(\xi_l)|^p \Big)^{q/p} \Big)^{1/q} < \infty, \qquad (6.3)$$

when $H = \Lambda_2 \cap \Gamma$. Also choose $\varphi, \kappa \in \mathcal{D}^{(s)}(X)$ such that $\varphi(x_0) \neq 0$ and

$$\kappa(x) \sum_{j \in J_{x_0}(\varepsilon)} \psi(x/\varepsilon - x_j) = 1, \quad \text{when} \quad x \in \operatorname{supp} \varphi.$$

Since $J_{x_0}(\varepsilon)$ is finite, Hölder's inequality gives

$$\begin{aligned}
|\varphi\, f|^{(D)}_{\mathcal{B}(H_0)} &= \Big| \sum_{j \in J_{x_0}(\varepsilon)} (\varphi\kappa)\,(f\, \psi(\,\cdot\,/\varepsilon - x_j)) \Big|^{(D)}_{\mathcal{B}(H_0)} \\
&\le \Big(\sum_{\xi_k \in H_0} \Big(\sum_{j \in J_{x_0}(\varepsilon)} |\mathscr{F}((\varphi\kappa) f\, \psi(\,\cdot\,/\varepsilon - x_j))(\xi_l)\omega(\xi_l)| \Big)^{q} \Big)^{1/q} \\
&\lesssim \Big(\sum_{\xi_l \in H_0} \Big(\sum_{j \in J_{x_0}(\varepsilon)} |\mathscr{F}((\varphi\kappa) f\, \psi(\,\cdot\,/\varepsilon - x_j))(\xi_l)\omega(\xi_l)|^p \Big)^{q/p} \Big)^{1/q},
\end{aligned}$$

where $H_0 = \Lambda_2 \cap \Gamma_0$. By Corollary 4.3 and (6.3) it now follows that the right-hand side in the last estimates is finite. Hence, $|\varphi\, f|^{(D)}_{\mathcal{B}(H_0)} < \infty$, which shows that $(x_0, \xi_0) \notin \mathrm{DF}_{\mathcal{B}}(f)$, and we have proved that $\mathrm{DF}_{\mathcal{B}}(f) \subseteq \mathrm{DF}_{\mathcal{C}}(f)$. The proof is complete. $\square$

In the following corollary we give a discrete description of the s-wave-front set, $\mathrm{WF}_s(f)$, from Section 2.

Corollary 6.4. *Let $q \in [1, \infty]$, $s > 1$, and let $\omega_k(\xi) \equiv e^{k|\xi|^{1/s}}$ for $\xi \in \mathbf{R}^d$ and $k > 0$. If $f \in (\mathcal{E}^{\{s\}})'(\mathbf{R}^d)$, then*

$$\bigcap_{k>0} \mathrm{DF}_{\mathscr{F}L^q_{(\omega_k)}}(f) = \mathrm{WF}_s(f). \qquad (6.4)$$

We remark that a discrete analogue of $\mathrm{WF}_s(f)$ also can be introduced in a similar way as in [26, 27]. Let us denote this set by $\mathrm{WF}_{s,T}(f)$, and refer to it as *toroidal* s-wave-front set. It can be proved that

$$\mathrm{WF}_{s,T}(f) = \mathbf{T}^d \times \mathbf{Z}^d \cap \mathrm{WF}_s(f), \tag{6.5}$$

where $\mathbf{T}^d$ is the torus in $\mathbf{R}^d$.

A significant difference between the toroidal wave-front sets and our discrete wave-front sets lies in the fact that $\mathrm{WF}_T(f)$ only informs about the *rational* directions for the propagation of singularities of f at a certain point, while $\mathrm{DF}(f) = \mathrm{WF}(f)$ takes care of *all* directional for f to that point, we refer to [16] for an example.

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